A Bayesian approach to estimating background flows from a passive scalar

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Advection-Diffusion Equation

Advection-diffusion equation describes behavior of a passive scalar $\theta$ driven by a background flow $v$ and diffusing with constant $\kappa$:

$$\dot{\theta} = -v \cdot \nabla \theta + \kappa \Delta \theta$$  \hspace{1cm} (1)

Example: Contaminant in a flowing fluid
Simple Vector Field
Effect of Simple Vector Field
Problem Statement

Given measurements of \( \theta \), e.g.

- **Spectral data:** \( y_j = \langle \theta(t), e_{k_j} \rangle_{L^2} + \eta_j(t) \)
- **Direct Observations:** \( v(x_j, t_k) + \eta_{j,k} \) ("weather forecasting" example)
- **Tracers:** \( \partial_t z_j = v(z_j, t); y_j = z_j(t_k) + \eta_{j,k} \) ("oceanography" example)

\[ \ldots \text{estimate } v \]
Bayesian Inference

- Assume a *prior* distribution - our “best guess” of $v$ before measurements.
- Incorporate measurements of passive scalar and provide an updated best guess, the *posterior* distribution.
- Scale of measurement noise $\eta$ determines how “strong” the data is
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Mechanics: Forward

Forward (“sample to data”) map, $\mathcal{G}$:

1. Given $\mathbf{v}$
2. Solve (1) for $\theta$
3. Measure, e.g. collect spectral components, measure at discrete points
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Mechanics: Inversion

Markov Chain Monte Carlo (MCMC) sampling of posterior distribution by acceptance probability related to potential (negative log likelihood) such that

\[
\frac{d\mu_y}{d\mu_0}(v) = \frac{1}{Z} \exp\left(-\Phi(v; y)\right)
\]  

For Gaussian measurement noise,

\[
\Phi(v; y) = \frac{1}{2} \left| \Gamma^{-1/2}(y - G(v)) \right|^2 - \frac{1}{2} \left| \Gamma^{-1/2}y \right|^2
\]
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A Toy Example

\[ \mathbf{v}_{true} \]

Starting vector field
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MCMC: Sampling
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MCMC: “Final” Vector Field

**V_{true}**

**Ending vector field**
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